

If You Are Uncertain About Uncertainty

Most physics students find the concept of uncertainty to be enigmatic; this does not need to be the case. Uncertainty is both an essential and useful component of experimental physics and is not as difficult to master as one may imagine. The following information has been assembled in an effort to help demystify the need for and implementation of uncertainty.

Part of the confusion stems from the fact that in early lab courses a large part of the emphasis is on developing the student's experimental proficiency. In order to facilitate this goal it is necessary to have a method of measuring the student's ability to perform an experiment precisely. To this end the student is often provided with an accepted or literature value for the experimental results. The success of the experiment is in part based on whether the student's experimental results are within a certain range of the accepted value. Although this approach and use of uncertainty is necessary during the development of students and their experimental skills it is not an appropriate approach for actual experiments. In an actual experiment the value determined through experimentation is the correct value. The theoretical value that is being tested may or may not be correct. Determining if the theoretical value is correct is often the purpose of the experiment itself.

In order to understand the appropriate use of uncertainty in experimental physics it is necessary to understand the roles played by theoretical and experimental physicists. The theorist working from previous experimental results and accepted concepts looks to expand the body of knowledge by exploring a certain aspect of the physical world. In this process certain assumptions are made, these assumptions are both conceptual and mathematical. As a result of these assumptions the final conclusion reached by the theorist may or may not be correct and needs to be tested.

The experimentalist's role is to test the theories developed by the theorist. The results provided by the experiment gives the theorist insight into the validity of the theory that has been developed. Although the experimentalist can never prove a theory correct, the experimental results can prove a theory wrong. With this role in mind the experimentalist must be honest about the limitations of the equipment and procedures employed during the experiment. These limitations are represented by the uncertainty associated with the final experimental results.

The method of uncertainty analysis employed for a particular experiment is an integral part of the experimental design. The experimenter chooses the most appropriate method of uncertainty analysis based on the conditions and types of limitations that may be encountered during the experiment. Great care needs to be taken during this process to ensure an honest final assessment of uncertainty for the experimental results.

The following information details some of the more common methods of uncertainty analysis. The methods discussed are by no means an exhaustive exploration of uncertainty but is rather a sampling of the methods that might be of most use to an undergraduate physics student participating in an undergraduate lab class.

0.1 Absolute Uncertainty

The easiest uncertainty concept to understand is that of absolute uncertainty. This is the direct uncertainty in a measurement or final value. Absolute uncertainty represents a range of values likely to enclose the true value for a given measurement or result. Imagine a student measuring a table with a meter stick; the meter stick is divided by marks that encompass one millimeter of distance. The most precise measurement that can be made in this case is plus or minus one millimeter. However if the student utilizes a large caliper with the capability of measuring distances to one micrometer then the uncertainty would be plus or minus one micrometer. Although the concept of measurement uncertainty is the same for each type of measurement made, the exact method to determine that uncertainty differs between types.

There are several means to represent absolute uncertainty. The following are two methods that can be used when writing a paper or report.

$$\begin{aligned} &\text{measurement} \pm \text{uncertainty (Units) ie.. } 9.812 \pm .002(m/s^2) \\ &\text{measurement(uncertainty) (Units) ie.. } 9.812(2)(m/s^2) \end{aligned}$$

The first method is self-explanatory, however; in the second method the number in parentheses is the uncertainty in the least significant figure of the measurement. This method is used extensively in tables or lists of literature values.

When adding or subtracting values with the same units the absolute uncertainty of the final value is the absolute uncertainties of the constituent values added in quadrature. This means that the final absolute uncertainty is the square root of the sum of the squares of the constituent absolute uncertainties. This method of adding in quadrature is very familiar to physics students, it is the same method used in the Pythagorean theorem.

$$\begin{aligned} &\text{If } d = a + b - c \\ &\text{then } \sigma_d = \sqrt{\sigma_a^2 + \sigma_b^2 + \sigma_c^2} \end{aligned}$$

0.2 Relative Uncertainty

The next type of uncertainty to be considered is relative uncertainty. Relative uncertainty, or percent uncertainty is the ratio of the size of the absolute uncertainty to the size of the measurement itself. Imagine the student with the meter stick again; if the student were to measure both the length of a room and a small coin, the one millimeter absolute uncertainty for the meter stick would have a much larger influence on the coins overall uncertainty. The relative uncertainty of a value is a useful indicator of the quality of that value. For example, a final value with a relative uncertainty of 1 percent is likely a more reliable value than one with a relative uncertainty of 25 percent.

To calculate the relative uncertainty for a measurement, divide the absolute uncertainty by the measured value and multiply by one hundred

$$\sigma_{ra} = \frac{\sigma_a}{a} (100).$$

This process can be used in reverse to determine the absolute uncertainty from the relative uncertainty by solving the above equation for the absolute uncertainty in a.

$$\sigma_a = a \frac{\sigma_{ra}}{100}$$

When multiplying or dividing values the relative uncertainty of the final value is the relative uncertainties of the constituent values added in quadrature

$$\text{If } d = \frac{ab}{c}$$

$$\text{Then } \frac{\sigma_{rd}}{100} = \frac{\sigma_d}{d} = \sqrt{\left(\frac{\sigma_a}{a}\right)^2 + \left(\frac{\sigma_b}{b}\right)^2 + \left(\frac{\sigma_c}{c}\right)^2}.$$

Note that when calculating the final absolute uncertainty for a value the multiplication and division of 100 can be omitted and the right two terms of the above equation used.

If the values in question involve exponents then the following form is used.

$$\text{If } d = \frac{a^l b^m}{c^n}$$

$$\text{Then } \frac{\sigma_{rd}}{100} = \frac{\sigma_d}{d} = \sqrt{\left(l \frac{\sigma_a}{a}\right)^2 + \left(m \frac{\sigma_b}{b}\right)^2 + \left(n \frac{\sigma_c}{c}\right)^2}$$

0.3 Multiple Measurements

0.3.1 Identical Absolute Uncertainties

At times it may be useful to determine the uncertainty in a measurement by identically repeating the measurement enough times to be able to calculate an average value for that measurement with a reasonable standard deviation. At this point, the uncertainty in any individual measurement is equal to the standard deviation of those measurements. However, when the values are averaged, then the mean measurement value has a smaller uncertainty, which is equal to the standard error of the mean, which is the standard deviation divided by the square root of the number of measurements

$$\sigma_{abs(x)} = \frac{\sigma_{stdx}}{\sqrt{N}}$$

this represents an absolute uncertainty in the average value.

As an example, imagine a student tasked with calculating the volume of a block of wood. The measurement device is a meter stick with an absolute uncertainty of .001 meters. The volume of the block can be calculated by multiplying the dimensions of the block together

$$v = lwh$$

and the relative uncertainty in the volume using one measurement for each dimension is the sum of the relative uncertainties of the dimensions added in quadrature.

$$\frac{\sigma_{rv}}{100} = \frac{\sigma_v}{v} = \sqrt{\left(\frac{\sigma_l}{l}\right)^2 + \left(\frac{\sigma_w}{w}\right)^2 + \left(\frac{\sigma_h}{h}\right)^2}.$$

In an effort to obtain the best estimation of the blocks volume all of the individual dimension measurements were repeated six times and the average values for each was used to calculate the volume. (Table. 1) contains the values and absolute uncertainties for the blocks measurements.

Table 1: Block Dimensions

Trial	length(m)	σ_l (m)	width(m)	σ_w (m)	height(m)	σ_h (m)
1	.1051	0.001	.0550	0.001	0.0223	0.001
2	.1053	0.001	.0551	0.001	0.0223	0.001
3	.1052	0.001	.0553	0.001	0.0223	0.001
4	.1049	0.001	.0549	0.001	0.0223	0.001
5	.1050	0.001	.0548	0.001	0.0223	0.001
6	.1048	0.001	.0552	0.001	0.0223	0.001

Since all of the absolute uncertainties for each measurement are the same, a simple average of each measurement is appropriate using the equation

$$\langle x \rangle = \frac{\sum x_i}{N}$$

where $\langle x \rangle$ is the measurement average, x is the individual measurement and i is the index of the measurement. The absolute uncertainty in the average value is

$$\sigma_{abs\langle x \rangle} = \frac{\sigma_{stdx}}{\sqrt{N}}$$

where $\sigma_{std\langle x \rangle}$ is the standard deviation of the measurements and N is the number of measurements. This provides an absolute uncertainty for the average value for a particular measurement that can be used to calculate the relative uncertainty for the volume of the block using

$$\frac{\sigma_{absv}}{v} = \sqrt{\left(\frac{\sigma_{abs\langle l \rangle}}{\langle l \rangle}\right)^2 + \left(\frac{\sigma_{abs\langle w \rangle}}{\langle w \rangle}\right)^2 + \left(\frac{\sigma_{abs\langle h \rangle}}{\langle h \rangle}\right)^2}$$

where σ_{absv} is the absolute uncertainty in the blocks volume.

0.3.2 Non Identical Absolute Uncertainties

There are times that the experimenter is faced with a situation where there are several independent values determined by experimentation for a particular measurement. If the values all have the same absolute uncertainty then the experimenter would simply use the method described in the previous section. However, if the individual values have different absolute uncertainties then the final average has to be weighted by those uncertainties. This process lends more weight to those measurements with the lowest uncertainty when calculating the average value.

As an example, X-ray diffraction is used to measure the d spacing of the individual atoms in a crystal lattice. The experiment bombards the surface of a crystal with high energy X-rays at a range of angles and measures the constructive and destructive interference patterns produced. This spacing can then be calculated using Bragg's Law

$$2d\sin\theta = n\lambda$$

where d is the lattice spacing, θ is the angle of constructive interference, n is the instance of constructive interference and λ is the wavelength of the X-ray source. Solving this equation for d

$$d = \frac{n\lambda}{2\sin\theta}$$

and using the angles provided by the experiment several values for d are calculated (Table. 2). Although the method of calculating the uncertainty in an equation with a trig function is covered in the next section, it is obvious that the uncertainty will be different for each of the six values for d .

Table 2: X-ray Diffraction data

Peak	$\lambda(\text{pm})$	$\theta(\text{rad})$	$\sigma_\theta(\text{rad})10^{-3}$	$d \text{ Spacing}(\text{m})$	$\sigma_{d\text{Spacing}}(\text{pm})$
1	63.09	0.1119	0.98	282.55	2.474
2	71.08	0.1262	0.99	282.39	2.232
3	63.09	0.2253	1.01	282.38	1.279
4	71.08	0.2543	1.03	282.55	1.156
5	63.09	0.3412	1.05	282.81	0.885
6	71.08	0.3866	1.06	282.79	0.799

The average value for d in this instance would be

$$\langle d \rangle = \frac{\sum \frac{d_i}{\sigma_i^2}}{\sum \frac{1}{\sigma_i^2}}$$

where $\langle d \rangle$ is the average value for d , d_i is the individual measurement and σ_i is the individual uncertainty in d_i . The denominator in the equation acts as a normalizing constant for the average.

The uncertainty in the value $\langle d \rangle$ would then be

$$\frac{1}{\sigma_{\langle d \rangle}} = \sqrt{\sum \frac{1}{\sigma_i^2}}$$

where $\sigma_{\langle d \rangle}$ is the uncertainty in the average value of d and σ_i are the individual uncertainties for each of the measurements used for the average.

0.4 Calculations Involving Trig Functions

Although this section specifically mentions trig functions the following method of uncertainty analysis can be applied to any equation used in data analysis. It is generally utilized in undergraduate lab courses when relative uncertainties cannot be used. In fact the method of relative uncertainty analysis detailed in this paper can be derived as a special case from the analysis covered in this section. The actual derivation will be left to a more comprehensive data analysis course.

The general equation used for uncertainty analysis is as follows. If the final value in question is a function of two variables with uncertainty $f(a, b)$ then the absolute uncertainty in that function is

$$\sigma_f = \sqrt{\left(\frac{\partial f}{\partial a}\right)^2 \sigma_a^2 + \left(\frac{\partial f}{\partial b}\right)^2 \sigma_b^2}$$

where $\left(\frac{\partial f}{\partial a}\right)$ is the partial derivative of the function in terms of a and σ_a is the absolute uncertainty in variable a . Variable b follows suit with additional variables added as needed. As mentioned this method can be used to evaluate the absolute uncertainty for a function in most undergraduate lab classes.

At this point it might be helpful to work through a specific problem; a good example of the use of this method is the Freshman Physics Vector Lab. The student is tasked with finding the Cartesian components of several forces at specific angles with uncertainty in both force and angle. The student then adds the Cartesian components of force to determine the resultant force and its uncertainty.

It is first necessary to determine the absolute uncertainty in both the force and angle before calculating the uncertainty in the components. The hanging mass exerts a force of $F = mg$; there is an uncertainty in the hanging mass and as a result there is an uncertainty associated with the force. This absolute uncertainty in the force can be calculated using relative uncertainty

$$\sigma_F = F \left(\frac{\sigma_m}{m} \right)$$

where σ_F is the absolute uncertainty in the force, m is the mass and σ_m is the absolute uncertainty in the mass (usually 1 gram).

The absolute uncertainty in the angle σ_θ is best determined by moving the pulley holder back and forth along the circumference of the apparatus and noting the angle of deviation that can be reached without affecting the center ring. This uncertainty is the result of the static coefficient of friction within the pulley itself. The pulley requires a certain amount of additional force before it will begin to move. With the two absolute uncertainties in hand the uncertainty in the Cartesian component of the force can be determined.

For the purposes of the Vector lab the following equation is used. If

$$F_x = F \cos(\theta),$$

$$\text{then } \sigma_{F_x} = \sqrt{\left(\frac{\partial F_x}{\partial F}\right)^2 \sigma_F^2 + \left(\frac{\partial F_x}{\partial \theta}\right)^2 \sigma_\theta^2}$$

$$\text{where } \left(\frac{\partial F_x}{\partial F}\right) = \cos(\theta)$$

$$\text{and } \left(\frac{\partial F_x}{\partial \theta}\right) = -F \sin(\theta).$$

Combining the equations together leads to the following. If $F_x = F \cos(\theta)$ then

$$\sigma_{F_x} = \sqrt{(\cos(\theta))^2 \sigma_F^2 + (-F \sin(\theta))^2 \sigma_\theta^2}$$

The same methodology can be applied to F_y resulting in the following. If $F_y = F \sin(\theta)$ then

$$\sigma_{F_y} = \sqrt{(\sin(\theta))^2 \sigma_F^2 + (F \cos(\theta))^2 \sigma_\theta^2}$$

If one were to assume that the absolute uncertainty in mass was equal to zero then the absolute uncertainty in the force is equal to zero and the previous two equations reduce to

$$\sigma_{F_x} = \sqrt{(-F \sin(\theta))^2 \sigma_\theta^2}$$

and

$$\sigma_{F_y} = \sqrt{(F \cos(\theta))^2 \sigma_\theta^2}$$

When adding the Cartesian components of the forces together the absolute uncertainty in the sum of the components is the sum of the constituent absolute uncertainties added in quadrature.

$$\begin{array}{l} \text{If } d = a + b - c \\ \text{then } \sigma_d = \sqrt{\sigma_a^2 + \sigma_b^2 + \sigma_c^2} \end{array}$$

At this point the student has the Cartesian components of the resultant force with the absolute uncertainties for each. The lab is completed by determining if the sum of all of the forces equals zero within the absolute uncertainty.

0.5 Summation

It is hoped that this paper has been successful in alleviating some of the confusion associated with uncertainty analysis. As previously stated it describes only a few of the more common methods of uncertainty analysis available. Although these methods should be sufficient for the average undergraduate lab experiment, the student should always be aware that a more appropriate method might be utilized.