1. Independent Pair of Fair 6-Sided Dice.
Consider an independent pair of fair 6-sided dice with sides indexed by $i$ and $j$.

a. Since they are independent, how does $p(i \mid j, I)$ relate to $p(i \mid I)$?

b. What is the probability of rolling $i=2$ on the first 6-sided die? That is, what is $p(i = 2 \mid I)$?

c. What quantity does $p(i = 2, j = 4 \mid I)$ represent? And what is its value?

d. What is the average value (also called the expected value) of $i$?

e. Is it possible to ever observe this expected value? Why or why not?

f. What is the expected value of $i+j$?

g. What is the most probable value of $i+j$?
2. Imagine that we have a pair of six-sided dice that are attached to one another with a string. Below is a partially-filled in table of the probabilities for rolling different values indexed by i and j.

| $P(i, j | I)$ | j = 1 | j = 2 | j = 3 | j = 4 | j = 5 | j = 6 |
|--------------|-------|-------|-------|-------|-------|-------|
| $i = 1$      | 0     | 0     | 0     | 0     | 0     | 0     |
| $i = 2$      | 0     | $\frac{2}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ | 0     |
| $i = 3$      | 0     | $\frac{4}{36}$ | $\frac{36}{36}$ | $\frac{2}{36}$ | 0     |
| $i = 4$      | 0     | $\frac{2}{36}$ | $\frac{36}{36}$ | $\frac{4}{36}$ | $\frac{2}{36}$ | 0     |
| $i = 5$      | 0     | $\frac{1}{36}$ | $\frac{36}{36}$ | $\frac{36}{36}$ | 0     |
| $i = 6$      | 0     | 0     | 0     | 0     | 0     | 0     |

a. Fill in the missing entry in the table above.

b. What is the expected value of $i+j$?

c. What is the probability $p(i | I)$ for all values of i?

\[
p(i = 1 | I) = \]
\[
p(i = 2 | I) = \]
\[
p(i = 3 | I) = \]
\[
p(i = 4 | I) = \]
\[
p(i = 5 | I) = \]
\[
p(i = 6 | I) = \]

d. What is the expected value of i?
3. In the problem #1, we found that \( p(i \mid j, I) = p(i \mid I) \) when \( i \) and \( j \) are independent.

Prove (algebraically using the sum and/or the product rules) that if
\[ p(x, y \mid I) = p(x \mid I) \, p(y \mid I) \]
then
\[ p(x \mid y, I) = p(x \mid I) \]
meaning that if \( x \) and \( y \) are independent, then you can just drop any conditioning of \( x \) on \( y \).

4. Rolling 6’s.
a. Imagine that you have one set of 6 six-sided dice. Calculate the probability that you will roll at least one 6 if you roll all six dice.

b. Consider that you have two sets of 6 six-sided dice (12 dice total). Calculate the probability that you will roll at least two 6’s if you roll all 12 dice.

c. Consider that you have three sets of 6 six-sided dice (18 dice total). Calculate the probability that you will roll at least three 6’s if you roll all 18 dice.

d. Are the probabilities in parts a-c equal? Why or why not? And how do these results compare to the simulations you performed in homework set HW1p?