

Measure, Probability, Quantum

Kevin H. Knuth¹ and John Skilling

1. University at Albany (SUNY), USA

A Question Posed

“Why is it that when I take two pencils and add one pencil, I always get three pencils?

And when I take two pennies and add one penny, I always get three pennies,

and so on with rocks and sticks and candy and monkeys and planets and stars.

Is this true by definition as in $2+1$ defines 3?

Or is it an experimental fact so that at some point in the distant past this observation needed to be verified again and again?”

- Kevin Knuth

A Question Posed

“I have tried, with little success, to get some of my friends to understand my amazement that the abstraction of integers for counting is both possible and useful.

Is it not remarkable that 6 sheep plus 7 sheep make 13 sheep; that 6 stones plus 7 stones make 13 stones? Is it not a miracle that the universe is so constructed that such a simple abstraction as a number is possible? To me this is one of the strongest examples of the unreasonable effectiveness of mathematics. Indeed, I find it both strange and unexplainable.”

- Hamming, 1980

A Question Posed



2 Pens



4 Pens

A Question Posed



2 Pens



4 Pens



2 Pens with 4 Pens

A Question Posed



2 Pens



4 Pens



2 Pens with 4 Pens

$$2 \oplus 4 \text{ Pens}$$

How does one determine the operator \oplus ?

A Question Posed

Knuth K.H. 2015.

FQXi 2015 Essay Contest Entry (**Third Place Prize**)

<http://fqxi.org/community/forum/topic/2382>

[arXiv:1504.06686](https://arxiv.org/abs/1504.06686) [math.HO]

The Deeper Roles of Mathematics in Physical Laws

Kevin H. Knuth

Departments of Physics and Informatics
University at Albany (SUNY), Albany NY, USA

“Familiarity breeds the illusion of understanding”
- Anonymous

Abstract

Many have wondered how mathematics, which appears to be the result of both human creativity and human discovery, can possibly exhibit the degree of success and seemingly-universal applicability to quantifying the physical world as exemplified by the laws of physics. In this essay, I claim that much of the utility of mathematics arises from our choice of description of the physical world coupled with our desire to quantify it. This will be demonstrated in a practical sense by considering one of the most fundamental concepts of mathematics: additivity. This example will be used to show how many physical laws can be derived as constraint equations enforcing relevant symmetries in a sense that is far more fundamental than commonly appreciated.

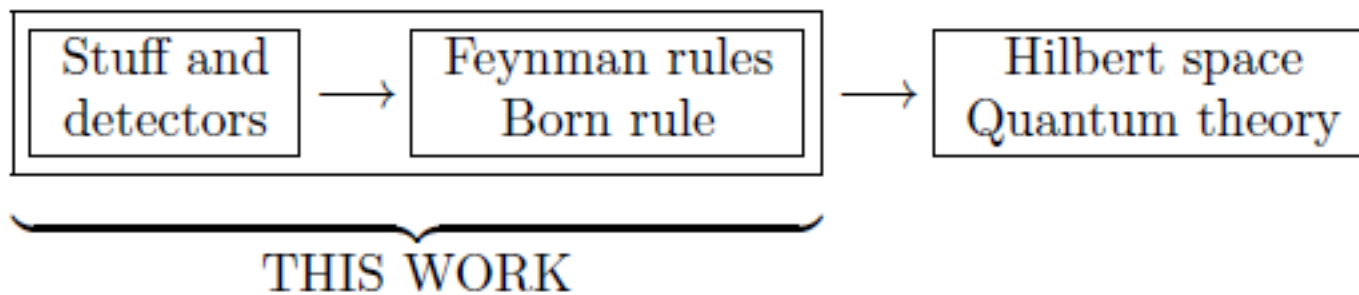
Introduction

Many have wondered how mathematics, which appears to be the result of both human creativity and human discovery, can possibly exhibit the degree of success and seemingly-universal applicability to quantifying the physical world as exemplified by the laws of physics (Wigner, 1960; Hamming, 1980). That is, if the laws of physics are taken as fundamental, then how can it be that mathematics, which is

“Measure that which is measurable and make measurable that which is not” --- Galileo

Our job is to make models of the world around us.

These models are often quantified by consistently assigning numbers so that the quantification captures relevant relationships and symmetries.



Combining Stuff: -with-

Closure

Given stuff **A** and stuff **B**, then combining stuff **A-with-B** is still stuff.

Commutativity

Order in which stuff is combined doesn't matter. **A-with-B** is the same as **B-with-A**

Associativity

Combination can be done in different but equivalent ways
(A-with-B)-with-C is the same as **A-with-(B-with-C)**

Reproducibility

We should be able to repeat experiments and have the results accumulate so that **A** is different from **A-with-A** is different from **A-with-A-with-A**, etc.

Vectors

Quantify stuff with a set of numbers (vector).

Stuff **A** is represented by **a**

Stuff **B** is represented by **b**

How to represent **A-with-B**?

Vectors

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Stuff **A** is represented by **a**
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How to represent **A-with-B**?

Does the operator **-with-** satisfy: ?

Closure

Commutativity

Associativity

Reproducibility

Vectors

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Stuff **A** is represented by **a**
Stuff **B** is represented by **b**

How to represent **A-with-B**?

Does the operator **-with-** satisfy: ?

Closure

Commutativity

Associativity

Reproducibility

If the operator **-with-** satisfies the above properties

Vector Sum Rule

A-with-B is represented by **a + b**
(up to isomorphism)

theorem

component-wise summation

Measures

Commensurability

If stuff has only one relevant property, then quantification requires only one dimension and our vector representation reduces to a scalar.

Measures

Commensurability

If stuff has only one relevant property, then quantification requires only one dimension and our vector representation reduces to a scalar.

If -with- satisfies the following properties:

- Closure
- Commutativity
- Reproducibility
- Associativity
- Commensurability

Measures

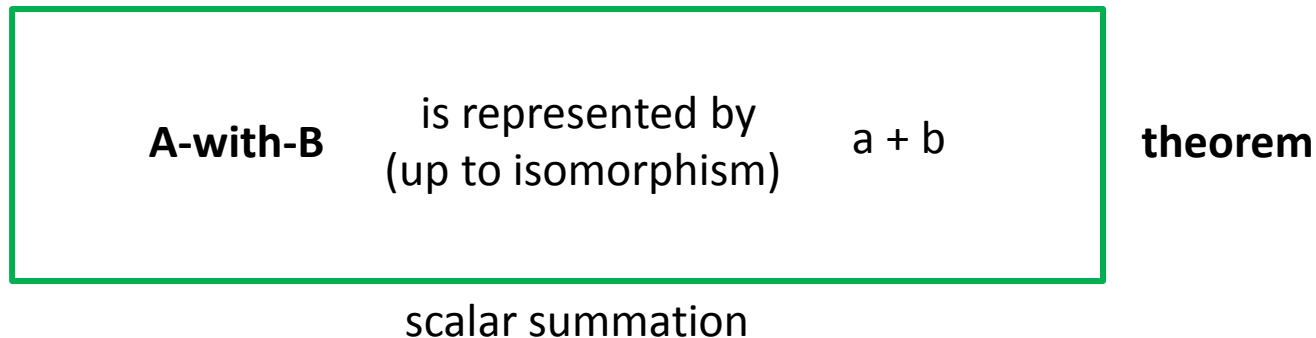
Commensurability

If stuff has only one relevant property, then quantification requires only one dimension and our vector representation reduces to a scalar.

If -with- satisfies the following properties:

- ☑ Closure
- ☑ Commutativity
- ☑ Reproducibility
- ☑ Associativity
- ☑ Commensurability

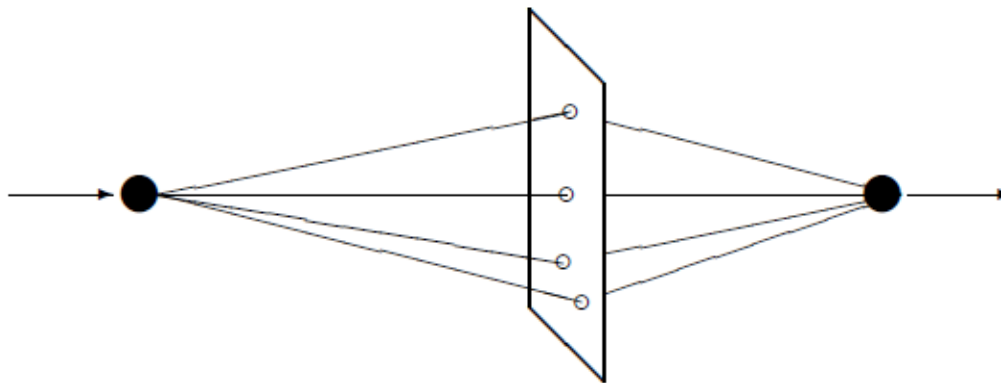
Measure Theory



This is **WHY** we add things when the five boxes above can be checked.

Paths

Consider a simple experiment consisting of a system that is injected from a source and is subsequently detected, such as in the example below of a particle interacting with a diffraction screen.



Here we aim to quantify paths from an initial state to a final state.

Detectors



with detector

$$p(\mathbf{u})$$

scalar quantification



without detector

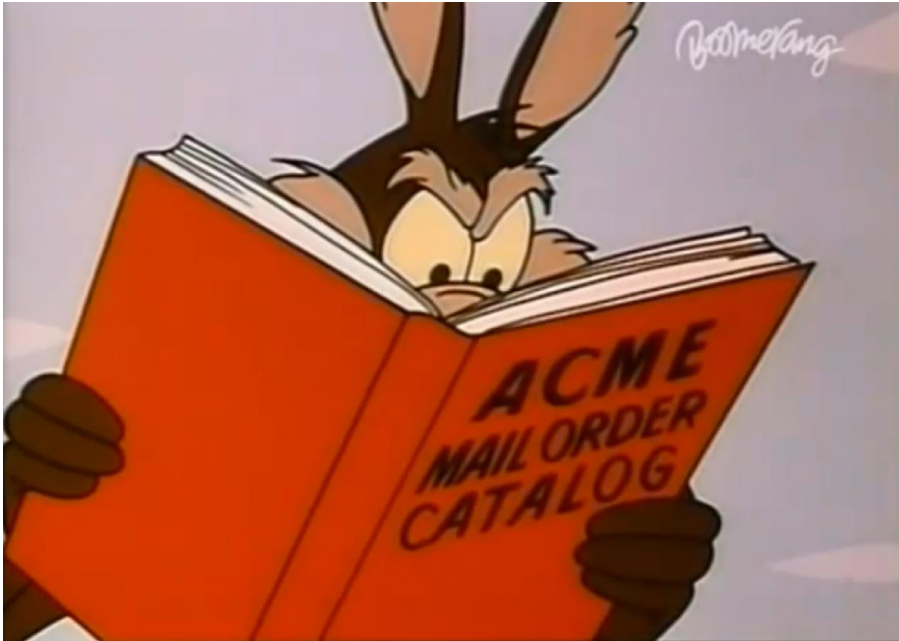
$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

pair quantification

The passage of an particle in a path with a detector will result in a bit of information, which can be promoted to a rate of passage (scalar).

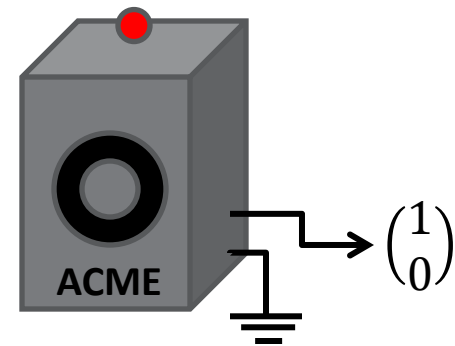
Our information about the particle involves at least **two** degrees of freedom, being some mix of the particle itself and whatever source(s) produced it (**Pair Postulate**).

Detectors



ACME Generic Detector

Model Number: 1/137



An all purpose detector that produces a pleasant 10 kHz **'ding'**, lights an LED indicator, and outputs a high bit when a particle is detected.

Paths Combined in Parallel

We can combine paths in parallel to form a new composite path.
Check the boxes:

- Closure** Yes, a composite path is a path
- Commutativity** Combine paths in any order with the same result
- Associativity** Grouping of paths gives same result
- Reproducibility** Yes, keep including more paths
- Commensurability** The vector dimension is subtle

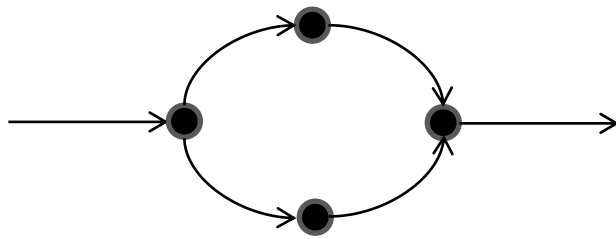
Thus when combining paths in parallel, the quantifications sum (additive).

Paths Combined in Parallel

Closure
Commutativity, Associativity
Reproducibility

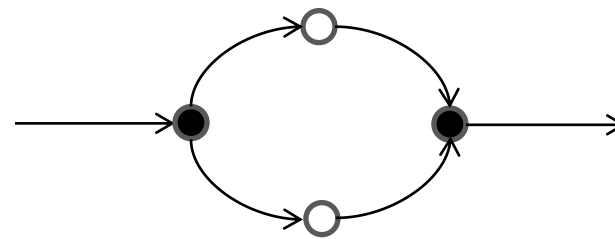


Additivity



$$p(\mathbf{u} \oplus \mathbf{v}) = p(\mathbf{u}) + p(\mathbf{v})$$

with detectors
scalar addition



$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$$

without detectors
component-wise addition

theorem

Paths Combined in Series: -then-

Tasks (paths / experiments) can be performed sequentially.

Closure

Task **A** followed by task **B**, **A-then-B** is also a task.

Distributivity

We relate **-then-** to **-with-**

(A-with-B)-then-C is the same as **(A-then-C)-with-(B-then-C)**

(A-then-B)-with-C is the same as **(A-with-C)-then-(B-with-C)**

Associativity

Combination can be done in different but equivalent ways

(A-then-B)-then-C is the same as **A-then-(B-then-C)**

A-then-B is represented by multiplication
(up to isomorphism)

theorem

Probability

Inferences *combine* using -or- (probspeak for -with-)



Closure



Commutativity



Associativity



Commensurability



Reproducibility

$$p(\text{A-or-B}) = p(\text{A}) + p(\text{B})$$

Sum Rule

Inferences *chain* using -from- (probspeak for reversed -then-)



Closure



Distributivity



Associativity

$$p(\text{A-from-C}) = p(\text{A-from-B}) \cdot p(\text{B-from-C})$$

Product Rule

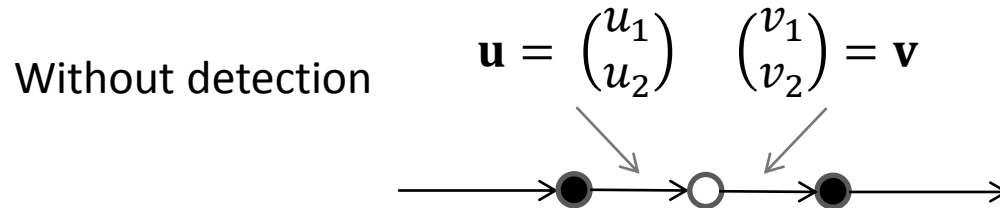
This is **WHY** we use probability here (and everywhere) when making inferences.

Bayes rules!

Paths Combined in Series : -then-

☑ Closure

☑ Distributivity



Bilinear Multiplication

$$\mathbf{u} \circ \mathbf{v} = \begin{pmatrix} \gamma_1 u_1 v_1 + \gamma_2 u_1 v_2 + \gamma_3 u_2 v_1 + \gamma_4 u_2 v_2 \\ \gamma_5 u_1 v_1 + \gamma_6 u_1 v_2 + \gamma_7 u_2 v_1 + \gamma_8 u_2 v_2 \end{pmatrix}$$

Paths Combined in Series : -then-

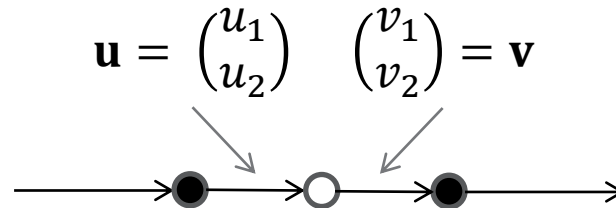


Closure



Distributivity

Without detection



Bilinear Multiplication

$$\mathbf{u} \circ \mathbf{v} = \begin{pmatrix} \gamma_1 u_1 v_1 + \gamma_2 u_1 v_2 + \gamma_3 u_2 v_1 + \gamma_4 u_2 v_2 \\ \gamma_5 u_1 v_1 + \gamma_6 u_1 v_2 + \gamma_7 u_2 v_1 + \gamma_8 u_2 v_2 \end{pmatrix}$$



Associativity plus freedom to shear results in five possible forms

$$\mathbf{u} \circ \mathbf{v} = \begin{pmatrix} u_1 v_1 - u_2 v_2 \\ u_1 v_2 + u_2 v_1 \end{pmatrix} \quad \text{or} \quad \mathbf{u} \circ \mathbf{v} = \begin{pmatrix} u_1 v_1 \\ u_1 v_2 + u_2 v_1 \end{pmatrix}$$

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Paths Combined in Series : -then-

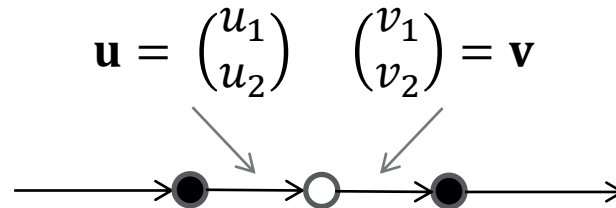


Closure



Distributivity

Without detection



Bilinear Multiplication

$$\mathbf{u} \circ \mathbf{v} = \begin{pmatrix} \gamma_1 u_1 v_1 + \gamma_2 u_1 v_2 + \gamma_3 u_2 v_1 + \gamma_4 u_2 v_2 \\ \gamma_5 u_1 v_1 + \gamma_6 u_1 v_2 + \gamma_7 u_2 v_1 + \gamma_8 u_2 v_2 \end{pmatrix}$$



Associativity plus freedom to shear results in five possible forms

$$\mathbf{u} \circ \mathbf{v} = \begin{pmatrix} u_1 v_1 - u_2 v_2 \\ u_1 v_2 + u_2 v_1 \end{pmatrix}$$

or

~~$$\mathbf{u} \circ \mathbf{v} = \begin{pmatrix} u_1 v_1 \\ u_1 v_2 + u_2 v_1 \end{pmatrix}$$~~

or

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or

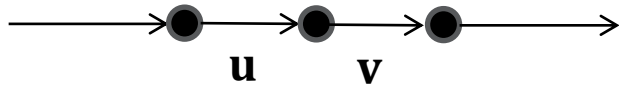
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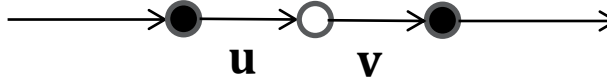
Product Rule

with detection



$$p(\mathbf{u} \bullet \mathbf{v}) = p(\mathbf{u})p(\mathbf{v})$$

without detection

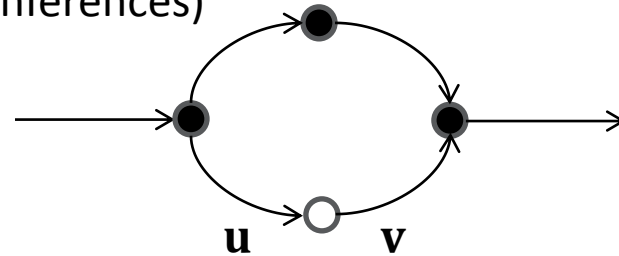
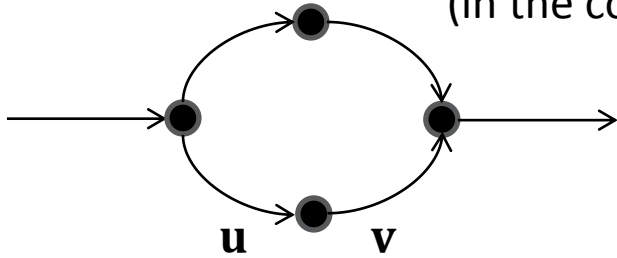


$$\mathbf{u} \circ \mathbf{v} = \begin{pmatrix} u_1 v_1 - u_2 v_2 \\ u_1 v_2 + u_2 v_1 \end{pmatrix}$$

theorem

Missing Detector

is equivalent to
(in the context of our inferences)



$$p(\mathbf{u} \bullet \mathbf{v}) = p(\mathbf{u} \circ \mathbf{v})$$

$$p\left(\begin{matrix} u_1 \\ u_2 \end{matrix}\right) p\left(\begin{matrix} v_1 \\ v_2 \end{matrix}\right) = p\left(\begin{matrix} u_1 v_1 - u_2 v_2 \\ u_1 v_2 + u_2 v_1 \end{matrix}\right)$$

Relating Pairs to Probabilities

$$p \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} p \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = p \begin{pmatrix} u_1 v_1 - u_2 v_2 \\ u_1 v_2 + u_2 v_1 \end{pmatrix}$$

Use polar coordinates

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \equiv r e^{i\theta}$$

solution

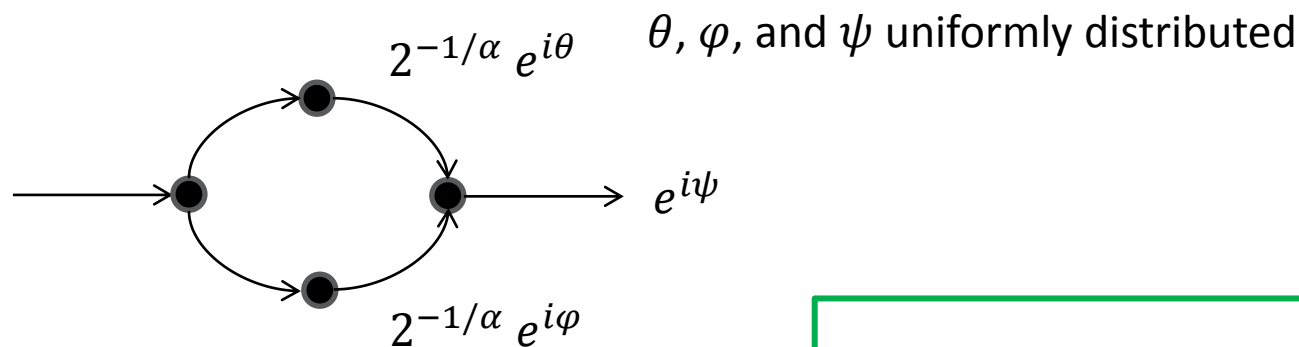
$$p(r e^{i\theta}) = r^\alpha e^{\beta\theta}$$

since θ is 2π periodic,
we have $r^\alpha e^{\beta\theta} = r^\alpha e^{\beta(\theta+2\pi)}$ implying $\beta = 0$

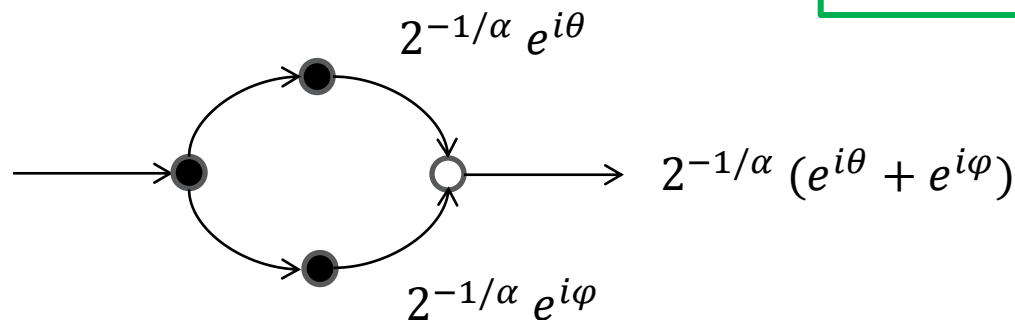
$$p(\mathbf{u}) = |\mathbf{u}|^\alpha$$

Relating Pairs to Probabilities

First consider



Next consider



θ, φ uniformly distributed

Both situations result
in probability one.

Relating Pairs to Probabilities

$2^{-\frac{1}{\alpha}} (e^{i\theta} + e^{i\varphi})$ with θ, φ uniformly distributed results in probability one.

$$\bar{p} = \langle p(\mathbf{u}) \rangle = \langle |\mathbf{u}|^\alpha \rangle = 1$$

$$\left\langle \frac{1}{2} |e^{i\theta} + e^{i\varphi}|^\alpha \right\rangle_{\theta, \varphi} = 1 \quad \text{which gives } \alpha = 2$$

so that

$$p(u) = |\mathbf{u}|^2 \quad \text{Born Rule}$$

Measure, Probability, Quantum

Measure Theory

$$A\text{-with-}B = a + b$$

theorems

Probability Theory

$$p(A\text{-or-}B) = p(A) + p(B)$$

$$p(A\text{-from-}C) = p(A\text{-from-}B) p(B\text{-from-}C)$$

Quantum Theory : Feynman Rules & Born Rule

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$$

$$\mathbf{u} \circ \mathbf{v} = \begin{pmatrix} u_1 v_1 - u_2 v_2 \\ u_1 v_2 + u_2 v_1 \end{pmatrix}$$

$$p(\mathbf{u}) = |\mathbf{u}|^2$$

“Quantum Mechanics will cease to look puzzling only when we will be able to derive the formalism of the theory from a set of simple physical assertions about the world.”

- Carlo Rovelli

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Closure

Commutativity

Associativity

Commensurability

Reproducibility

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$$

Closure

Distributivity

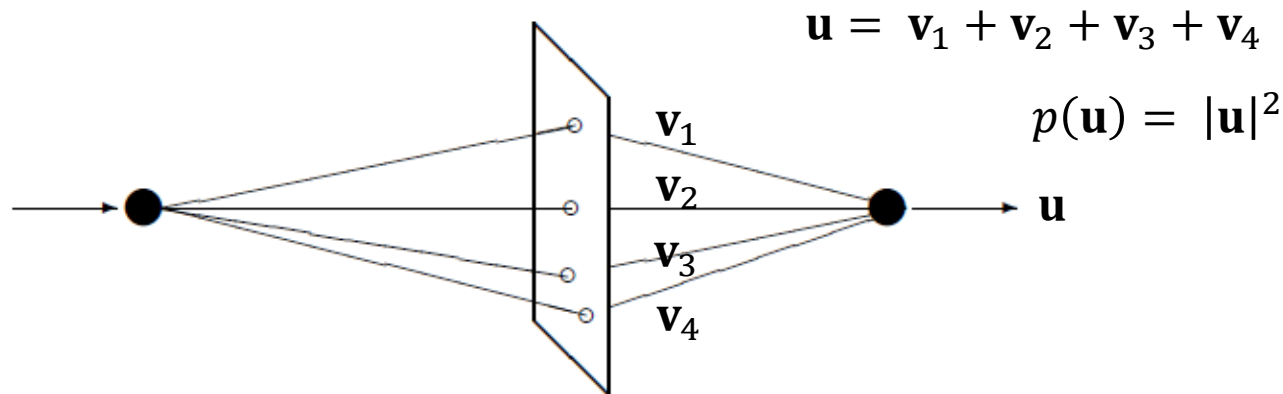
Associativity

$$\mathbf{u} \circ \mathbf{v} = \begin{pmatrix} u_1 v_1 - u_2 v_2 \\ u_1 v_2 + u_2 v_1 \end{pmatrix}$$

$$p(\mathbf{u}) = |\mathbf{u}|^2$$

Quantum Mechanics

We use the pattern of paths to construct a corresponding pattern of partially-known complex amplitudes and associated probabilities, as dictated by simple symmetries.



We then use standard probabilistic inference to restrict those possibilities in accordance with observations, thereby gaining predictive power over what our detectors will record.

That is what Quantum Mechanics is.