Distinguishability: The Lesson of the Two Children Problem

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Information Increasing
Distinguishability Increasing →

$$P(\boldsymbol{BQ}, BU \mid \mathbf{BQ}SU \vee \boldsymbol{SU}BQ, I) = \frac{\frac{2}{p} - \frac{1}{p^2}}{\frac{4}{p} - \frac{1}{p^2}}$$

As the probability, p, of the quality Q decreases, the two children become more distinguishable. The probability that they are both boys goes from 1/3 to 1/2. (see below for details)

Two Children
At Least one is a Boy
(no qualifications, $p = 1$)

$$P(\text{both Boys}) = \frac{\frac{2}{p} - \frac{1}{p^2}}{\frac{4}{p} - \frac{1}{p^2}} = \frac{2-1}{4-1} = \frac{1}{3}$$

Two Children
At Least one is a Boy born on Tuesday ($p = 1/7$)

$$P(\text{both Boys}) = \frac{\frac{2}{p} - \frac{1}{p^2}}{\frac{4}{p} - \frac{1}{p^2}} = \frac{13}{27}$$

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Two Children (complete specification)
At Least one is my son Henry ($\lim p \to 0$)
The children can be completely distinguished

$$P(\text{both Boys}) = \frac{\frac{2}{p} - \frac{1}{p^2}}{\frac{4}{p} - \frac{1}{p^2}} \to \frac{1}{2}$$

and can be treated independently.

Introduction

Puzzles designed with an intellectual twist act as an entertaining way of presenting a lesson that can highlight a deficit in our thinking or enlighten us in an often surprising way. However, it seems that the more subtle the lesson embedded within the puzzle, the more disagreement and debate ensues.

Here I consider the Two Child Problem where one makes inferences about the sexes of two children based on a spectrum of prior information:
(May 24, 2010 issue of New Scientist magazine)

I have two children.
One is a boy born on Tuesday.
What is the probability that both children are boys?

At first glance it seems to be a simple problem, since how is it be possible that the information about the day of a child's birth provides any information about the sex of another child?

I demonstrate that the additional information, being born on a Tuesday, does affect one's inferences. Certainly the day on which one child is born cannot possibly affect the sex of another child, but it can affect what you know about the sex of another child!
How?
Read the poster.

This problem demonstrates that weird counter-intuitive correlations can appear in classical inference problems. Understanding these may help in understanding weird quantum correlations, since quantum mechanics is an inferential theory.

Boy Scout Database

LEAST INFORMATIVE SITUATION
Consider a Boy Scout Database of families each of whom has at least one child who is a boy. A similar question to that posed in the *Introduction* is:

Select a family at random that has two children at least one of whom is a boy.
What is the probability that both children are boys?

Bold Font is Child #1, Normal Font is Child #2
B = Boy, $S = G \vee B$ (Girl or Boy)

Bayes

At least one child is a Boy

$$P(\boldsymbol{B}B \mid \boldsymbol{B}S \vee \boldsymbol{S}B, I) = \frac{1}{Z} \mathrm{P}(\boldsymbol{B}B \mid I)\, P(\boldsymbol{B}S \vee \boldsymbol{S}B \mid \boldsymbol{B}B)$$

$$P(\boldsymbol{B}B \mid I) = \frac{1}{4}$$

$$\begin{aligned} P(\boldsymbol{B}S \vee \boldsymbol{S}B \mid \boldsymbol{B}B) &= \mathrm{P}(\boldsymbol{B}S \mid \boldsymbol{B}B) + P(\boldsymbol{S}B \mid \boldsymbol{B}B) - P(\boldsymbol{B}S \wedge \boldsymbol{S}B \mid \boldsymbol{B}B) \\ &= \mathrm{P}(\boldsymbol{B}S \mid \boldsymbol{B}B) + P(\boldsymbol{S}B \mid \boldsymbol{B}B) - P(\boldsymbol{B}B \mid \boldsymbol{B}B) \\ &= 1 + 1 - 1 = 1 \end{aligned}$$

$$\begin{aligned} Z = {} & \mathrm{P}(\boldsymbol{B}B \mid I)\, P(\boldsymbol{B}S \vee \boldsymbol{S}B \mid \boldsymbol{B}B) + \\ & \mathrm{P}(\boldsymbol{G}B \mid I)\, P(\boldsymbol{G}S \vee \boldsymbol{S}B \mid \boldsymbol{G}B) + \\ & \mathrm{P}(\boldsymbol{B}G \mid I)\, P(\boldsymbol{B}S \vee \boldsymbol{S}G \mid \boldsymbol{B}G) \\ = {} & \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 1 = \frac{3}{4} \end{aligned}$$

So that

$$P(\boldsymbol{B}B \mid \boldsymbol{B}S \vee \boldsymbol{S}B, I) = 1/3.$$

Equiprobable Cases:
$\boldsymbol{B}B \quad \boldsymbol{B}G \quad \boldsymbol{G}B \quad \cancel{\boldsymbol{G}G}$

More General Case

One may possess additional information, a quality Q, possessed by one of the children with prior probability p.

There is a family with two children at least one of whom is a boy with the quality Q (eg. born on Tuesday). Let U represent an unspecified quality (eg. an unspecified day of the week).

Bayes

$$\begin{aligned} & P(\boldsymbol{BQ}BU \mid \mathbf{BQ}SU \vee \boldsymbol{SU}BQ, I) \\ & = \frac{1}{Z} \mathrm{P}(\boldsymbol{BQ}BU \mid I)\, P(\boldsymbol{BQ}SU \vee \boldsymbol{SU}BQ \mid \boldsymbol{B}B) \end{aligned}$$

The likelihood has a term that subtracts off the possibility that both children have the quality Q. This is the degree to which the children can be distinguished:

$$\begin{aligned} P(\boldsymbol{BQ}SU \vee \boldsymbol{SU}BQ \mid \boldsymbol{B}B) = {} & \mathrm{P}(\boldsymbol{BQ}SU \mid \boldsymbol{B}B) + P(\boldsymbol{SU}BQ \mid \boldsymbol{B}B) + \\ & -P(\boldsymbol{BQ}SU \wedge \boldsymbol{SU}BQ \mid \boldsymbol{B}B) \\ = {} & \mathrm{P}(\boldsymbol{BQ}SU \mid \boldsymbol{B}B) + P(\boldsymbol{SU}BQ \mid \boldsymbol{B}B) - P(\boldsymbol{BQ}BQ \mid BB) \end{aligned}$$

The general solution is

$$P(\boldsymbol{BQ}BU \mid \boldsymbol{BQ}SU \vee \boldsymbol{SU}BQ, I) = \frac{\frac{1}{p} + \frac{1}{p} - \frac{1}{p^2}}{\frac{2p}{(2p)^2} + \frac{2p}{(2p)^2} - \frac{1}{(2p)^2}} = \frac{\frac{2}{p} - \frac{1}{p^2}}{\frac{4}{p} - \frac{1}{p^2}}$$

The quality Q serves to distinguish the children, while appearing to introduce unexpected inferential correlations.

Implications for understanding Quantum Mechanics?