Bayesian Data Analysis
A PHY 451/551, I CSI 451/551, and I INF 451/551
HW#5w Show all work

Lighthouse Problem (Written component)
Here we will consider the lighthouse problem, which was discussed in class as well as in the text Sivia and Skilling. Consider that a lighthouse is located at some distance along the coastline and some distance offshore. The lighthouse rotates at a constant rate and can fire a pencil thin beam of light at random in any direction (uniformly distributed). The lighthouse flashes hit the coast 64 times resulting in detections at the following locations:
D = [4.73, 0.45, -1.73, 1.09, 2.19, 0.12, 1.31, 1.00, 1.32, 1.07, 0.86, -0.49, -2.59, 1.73, 2.11, 1.61, … 4.98, 1.71, 2.23, -57.20, 0.96, 1.25, -1.56, 2.45, 1.19, 2.17, -10.66, 1.91, -4.16, 1.92, 0.10, 1.98, ...
-2.51, 5.55, -0.47, 1.91, 0.95, -0.78, -0.84, 1.72, -0.01, 1.48, 2.70, 1.21, 4.41, -4.79, 1.33, 0.81, …
0.20, 1.58, 1.29, 16.19, 2.75, -2.38, -1.79, 6.50, -18.53, 0.72, 0.94, 3.64, 1.94, -0.11, 1.57, 0.57];

1. The orientation of the lighthouse is described by the angle θ such that θ = 0 is aimed directly at (perpendicular to) the coastline. Similarly, directions $\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$ are oriented parallel to the coastline.

a. Consider that the beam can hit a point along the shore ($-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$) with uniform probability. What is the normalized probability density for the angle, $P(\theta \mid I)$, in this case?

b. Adopt a model in which the lighthouse is located at position (x_0, y_0). Perform a change of variables to find the probability density of the beam hitting at position x along the shore. This is the likelihood $P(x \mid x_0,\ y_0,\ I)$ of detecting the beam at position x given the model parameters (x_0, y_0) describing the lighthouse location.

c. Write the joint likelihood for the $N = 64$ light flash positions $\{x_1,\ x_2,\ \cdots,\ x_N\}$.

d. Write Bayes' Theorem for the posterior probability $P(x_0, y_0 \mid \{x_1,\ x_2,\ \cdots,\ x_N\}, I)$ writing the evidence simply as the constant Z.

2. a. Write the logarithm of the posterior probability, $\log P$, found in 1d.

b. Find the first derivative of $\log P$ with respect to both parameter values, x_0 and y_0.

c. Find the second derivatives: $\frac{\partial^2}{\partial x_0^2}\log P$, $\frac{\partial^2}{\partial y_0^2}\log P$, $\frac{\partial^2}{\partial x_0\,\partial y_0}\log P$.

3. Show that by setting the first derivatives equal to zero, that it is not possible to analytically solve for the most probable lighthouse position $(\widehat{x_0}, \widehat{y_0})$.

4. We will now attempt to find a numerical solution while taking advantage of the analytical expressions for the second derivatives to obtain an analytical estimate of the uncertainty.

a. Use the function `fminsearch` in Matlab to find the most probable position, $(\widehat{x_0}, \widehat{y_0})$, of the lighthouse by finding the maximum of the log posterior $\log P$.

b. Evaluate the second derivatives at the most probable position $(\widehat{x_0}, \widehat{y_0})$, and use these to find the uncertainties in the position estimates writing the final solution as $\widehat{x_0} \pm \sigma_{x_0}$ and $\widehat{y_0} \pm \sigma_{y_0}$.