Measuring the Area of a Circle

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1 Introduction

These notes address the problem of estimating the area of a circle, $s$, by measuring its radius. We assume that the reader has found the most probable radius of the circle, $r_0$, as well as the uncertainty in that estimate, $\sigma_r$, and that the posterior probability is Gaussian in form

$$P(r|\text{data}, I) \propto \exp \left[ -\frac{1}{2\sigma_r^2} (r - r_0)^2 \right].$$

(1)

If the reader is in a situation where he/she has instead found the most probable diameter of the circle, $d_0$, as well as the uncertainty in that estimate $\sigma_d$, then after working carefully through the following material, consider a slightly simpler problem of obtaining an estimate of the radius and its uncertainty from the estimated diameter and its uncertainty.

This problem is a change in variables, similar to the one considered below, where one transforms a diameter to a radius where $r = \frac{1}{2}d$. It is a recommended exercise to show that the most probable radius $r_0$ given the most probable diameter $d_0$ is given by

$$r_0 = \frac{1}{2}d_0.$$  

(2)

One should also work through the second derivative to show that the uncertainties are related by

$$\sigma_r = \frac{1}{\sqrt{2}}\sigma_d.$$  

(3)

Thus, one can report either a radius of $r_0 \pm \sigma_r$ or a diameter of $d_0 \pm \sigma_d$ where $r_0$, $d_0$, $\sigma_r$ and $\sigma_d$ are related by (2) and (3).

2 Changing Variables

We treat the problem as a change of variables where the area of the circle $s$ is a function of the radius $r$ by the usual $s = \pi r^2$. We begin by noting that the probability that a circle has a radius between $r$ and $r + dr$ must be equal to the probability that the circle has an area between $s$ and $s + ds$ where $s = \pi r^2$. Since the transformation from $r$ to $s$ does not involve a sign change, we can write

$$P(s|\text{data}, I)ds = P(r|\text{data}, I)dr$$  

(4)

so that

$$P(s|\text{data}, I) = P(r|\text{data}, I) \left( \frac{ds}{dr} \right)^{-1}.$$  

(5)
Evaluating the derivative 
\[ \frac{ds}{dr} = 2\pi r \]
we can then write
\[ P(s|\text{data}, I) = \frac{1}{2\pi r} P(r|\text{data}, I). \] (6)

Given that \( P(r|\text{data}, I) \) is a Gaussian \( \mathcal{N} \), we can write
\[ P(s|\text{data}, I) = \frac{1}{2\pi r} e^{-\frac{1}{2\pi r^2} (r-r_0)^2}, \] (7)
which by substituting \( r = \sqrt{s} \) can be written as
\[ P(s|\text{data}, I) = \frac{1}{2\sqrt{s}} e^{-\frac{1}{2\pi s} (s^{\frac{1}{2}} - \pi^{\frac{1}{2}} r_0)^2}. \] (8)

3 Gaussian Approximation

It is clear that the probability \( P(s|\text{data}, I) \) in (8) is not a Gaussian function. In this section we will find the optimal value of the area \( s_0 \) and approximate the posterior probability of \( s \) with a Gaussian so that we can approximate the uncertainty in our estimate \( s_0 \) with the standard deviation of the Gaussian approximation.

To approximate \( P(s|\text{data}, I) \) with a Gaussian, we begin by taking the natural logarithm of the posterior probability by writing
\[ L(s) = \ln[P(s|\text{data}, I)] \]
\[ = \ln \left[ \frac{1}{2\sqrt{s}} e^{-\frac{1}{2\pi s} (s^{\frac{1}{2}} - \pi^{\frac{1}{2}} r_0)^2} \right] \]
\[ = -\frac{1}{2\pi s} \left( s^{\frac{1}{2}} - \pi^{\frac{1}{2}} r_0 \right)^2 - \frac{1}{2} \ln s - \ln(2\sqrt{\pi}) \] (9)

The idea is to perform a Taylor series expansion so that
\[ L(s) \approx L(s_0) + \frac{dL(s)}{ds} \bigg|_{s=s_0} (s-s_0) + \frac{1}{2} \left[ \frac{d^2 L(s)}{ds^2} \right]_{s=s_0} (s-s_0)^2 + \ldots \] (10)

At the most probable value \( s_0 \) of the area, the first derivative \( \left. \frac{dL(s)}{ds} \right|_{s=s_0} = 0 \). This will leave us with a log probability with a quadratic dependence on \( s \), which when exponentiating will leave us with a Gaussian distribution who's uncertainty will depend on the inverse of the curvature of the log probability at the optimal value \( s_0 \):
\[ P(s|\text{data}, I) \approx Ae^{-\frac{1}{2\sigma_s^2} (s-s_0)^2} \] (11)

where \( A = e^{L(s_0)} \) and
\[ \sigma_s = \left( -\left[ \frac{d^2 L(s)}{ds^2} \right]_{s=s_0} \right)^{-\frac{1}{2}} \] (12)
so that
\[ P(s|\text{data}, I) \approx Ae^{-\frac{1}{2\sigma_s^2} (s-s_0)^2} \] (13)
3.1 Estimating the Area

First, we find the first derivative

\[
\frac{dL(s)}{ds} = - \frac{1}{2\pi \sigma_r^2} 2 \left( s^{\frac{1}{2}} - \pi^{\frac{1}{2}} r_0 \right) \frac{1}{2} s^{-\frac{1}{2}} - \frac{1}{2s}
\]

evaluate it at \( s_0 \) and set it to zero to solve for \( s_0 \) so that

\[
- \frac{1}{2\pi \sigma_r^2} 2 \left( \sqrt{s_0} - \sqrt{\pi r_0} \right) \frac{1}{2} s_0^{-\frac{1}{2}} = \frac{1}{2s_0}
\]

and

\[
s_0 - \sqrt{\pi r_0} \sqrt{s_0} = -\pi \sigma_r^2.
\]

Writing \( t = \sqrt{s_0} \), we have that

\[
t^2 - \sqrt{\pi r_0} t + \pi \sigma_r^2 = 0.
\]

The solution is found by the quadratic formula

\[
t = \frac{\sqrt{\pi r_0} \pm \sqrt{\pi r_0^2 - 4\pi \sigma_r^2}}{2},
\]

so that \( s_0 \) is found from the positive solution

\[
s_0 = \frac{1}{2} \left[ \pi r_0^2 - 2\pi \sigma_r^2 + \sqrt{\pi^2 r_0^4 - 4\pi r_0^2 \sigma_r^2} \right]. \tag{14}
\]

Note that the uncertainty in the estimated radius \( \sigma_r \) affects the best estimate of the area. If we let this uncertainty go to zero, \( \sigma_r = 0 \), then we have the expected result

\[
s_0 = \pi r_0^2,
\]

but this holds ONLY IF the uncertainty in the radius estimate is zero! In practice, one must use (14) to find the best estimate of the area of the circle.

3.2 Estimating the Uncertainty in the Area Estimate

We now compute the second derivative of the log probability

\[
\frac{d^2}{ds^2} L(s) = \frac{d}{ds} \left[ \frac{dL(s)}{ds} \right]
\]

\[
= \frac{d}{ds} \left[ \frac{1}{2\pi \sigma_r^2} 2 \left( s^{\frac{1}{2}} - \pi^{\frac{1}{2}} r_0 \right) \frac{1}{2} s^{-\frac{1}{2}} - \frac{1}{2s} \right]
\]

\[
= \frac{d}{ds} \left[ \frac{1}{2\pi \sigma_r^2} \left( 1 - \pi^{\frac{1}{2}} r_0 s^{-\frac{1}{2}} \right) - \frac{1}{2s} \right]
\]

\[
= - \frac{r_0}{4\pi \sigma_r^2 s_0^{\frac{3}{2}}} + \frac{1}{2s^2}. \tag{15}
\]

The second derivative above is evaluated at the peak \( s = s_0 \) which gives

\[
\left. \frac{d^2}{ds^2} L(s) \right|_{s=s_0} = - \frac{r_0}{4\pi \sigma_r^2 s_0^{\frac{3}{2}}} + \frac{1}{2s_0^2}. \tag{16}
\]
Finally, the uncertainty in the estimated area is given by (12), which is

\[
\sigma_s = \left( \frac{r_0}{4\pi \sigma_{s_0}^2} - \frac{1}{2s_0^2} \right)^{-\frac{1}{2}}.
\]  

(17)

As a check, the units should be those of an area, which they are.

The final estimate of the area can then be written as \( s = s_0 \pm \sigma_s \) where \( s_0 \) and \( \sigma_s \) are given by (14) and (17).